Quiz 5 – Solutions

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1. Complete[*] the partial sentence below into a precise definition for, or a precise mathematical characterization of, the italicized term:

   Suppose $m, n \in \mathbb{Z}_{>0}$. A *linear transformation* $T : \mathbb{R}^m \to \mathbb{R}^n$ is ...

   Solution: A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a *linear transformation* if, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$,
   $$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \quad \text{and} \quad T(\alpha\,\mathbf{u}) = \alpha\,T(\mathbf{u}).$$
   Equivalently, for all $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$,
   $$T(a\mathbf{u} + b\mathbf{v}) = a\,T(\mathbf{u}) + b\,T(\mathbf{v}).$$

2. Let L be a line through the origin in $\mathbb{R}^2$. Consider the mapping
   $$\mathbb{R}^2 \xrightarrow{\pi_L} \mathbb{R}^2 \qquad \vec{x} \mapsto \text{“the projection of } \vec{x} \text{ onto } L.\text{”}$$

   (a) Write a formula for $\pi_L(\vec{x})$ using the dot product and a unit vector $\vec{u}$ in the direction of L.

   Solution: If $\vec{u}$ is any unit vector pointing along L (so $\|\vec{u}\| = 1$), then the orthogonal projection of $\vec{x}$ onto L is
   $$\pi_L(\vec{x}) = (\vec{x} \cdot \vec{u})\,\vec{u}.$$

   (b) *Using the definition of linear transformation,* prove that π_L is a linear transformation.

   Solution: Let $\vec{u}$ be as above with $\|\vec{u}\| = 1$. For any $\vec{x}, \vec{y} \in \mathbb{R}^2$ and any $\alpha \in \mathbb{R}$,
   $$\pi_L(\vec{x} + \vec{y}) = \big((\vec{x} + \vec{y}) \cdot \vec{u}\big)\,\vec{u} = \big(\vec{x} \cdot \vec{u} + \vec{y} \cdot \vec{u}\big)\,\vec{u} = (\vec{x} \cdot \vec{u})\,\vec{u} + (\vec{y} \cdot \vec{u})\,\vec{u} = \pi_L(\vec{x}) + \pi_L(\vec{y}),$$
   and
   $$\pi_L(\alpha\vec{x}) = \big((\alpha\vec{x}) \cdot \vec{u}\big)\,\vec{u} = \alpha(\vec{x} \cdot \vec{u})\,\vec{u} = \alpha\,\pi_L(\vec{x}).$$
   We used bilinearity of the dot product in each step. Hence π_L satisfies additivity and homogeneity, so π_L is linear.

[*]For full credit, please write out fully what you mean instead of using shorthand phrases.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Let $\mathcal{I}([0,1])$ denote the set of integrable functions on $[0,1]$. The map $I\colon \mathcal{I}([0,1]) \to \mathbb{R}$ defined by
   $$I(f) = \int_0^1 f(x)\,dx$$
   is a linear transformation.

   Solution: TRUE. For any $f, g \in \mathcal{I}([0,1])$ and any $a, b \in \mathbb{R}$, we have $af + bg \in \mathcal{I}([0,1])$ and
   $$I(af+bg) = \int_0^1 \big(af(x) + bg(x)\big)\,dx = a\int_0^1 f(x)\,dx + b\int_0^1 g(x)\,dx = a\,I(f) + b\,I(g).$$
   This uses linearity of the (Riemann or Lebesgue) integral. Therefore I is linear.